Spectral Modes Viewpoint of H, HD+DH, and HD-DH

Intuitive View

The Walsh–Hadamard matrix H is not just a mixer. It naturally splits space into two kinds of global patterns:

½(I+H) and ½(I-H)

These are the +1 and -1 eigenspaces of H.

Very intuitively:

- I+H selects patterns that are globally cooperative or self-consistent under Hadamard mixing.
- I-H selects patterns that are globally oppositional or sign-flipping under Hadamard mixing.

So the system already contains a built-in binary spectral structure before D is introduced.

The diagonal matrix D introduces a second binary structure:

$d_i \in \{+1,-1\}$

which partitions channels into two local groups.

The operators:

HD+DH and HD-DH

then describe how:

- local binary structure (D)
- interacts with
- global binary spectral structure (I ± H).

A useful intuition is:

object	meaning
D	local sign partition
I+H	cooperative global modes
I-H	oppositional global modes

Then:

- HD+DH reinforces same-sign interactions through spectral modes.
- HD-DH transfers information across opposite-sign groups through spectral modes.

So the system behaves like a two-layer binary geometry:

1. local agreement/disagreement
2. global spectral agreement/disagreement

and the interesting behavior comes from how those two binary structures overlap.

You can think of spectral modes as the system's natural global patterns of behavior — the large-scale shapes or oscillations that remain structurally stable under the transform H.

Technical Notes / Research Directions

Basic spectral decomposition

Since:

$H^2=I$

the eigenvalues of H are:

± 1

with orthogonal projectors:

$P_+ = \frac{1}{2}(I+H)$

$P_- = \frac{1}{2}(I-H)$

and:

$H=P_+-P_-$

Reinterpretation of HD ± DH

Using:

$H=P_+-P_-$

gives:

HD ± DH

$(P_+-P_-)D \pm D(P_+-P_-)$

revealing interactions between:

- local binary sectors (from D)
- spectral parity sectors (from $P_\pm$).

Two-level binary grading

The system naturally induces a:

$$\mathbb{Z}_2 \times \mathbb{Z}_2$$

grading structure:

local sector	**spectral sector**
same sign	cooperative mode
same sign	oppositional mode
opposite sign	cooperative mode
opposite sign	oppositional mode

Spectral transfer interpretation

The commutator:

$[H,D]=HD-DH$

can be interpreted as:

- an eigenspace transfer operator,

- measuring how D couples the +1 and -) eigenspaces of H.

If:

$[H,D]=0$

then the spectral sectors remain decoupled.

Spectral clustering interpretation

- HD+DH:
  preserves interactions inside local sign sectors.
- HD-DH:
  isolates interactions across local sign sectors.

Combined with the $P_{\pm}$ decomposition, this produces a layered signed graph structure with spectral parity labels.

Possible connections

Potentially related mathematical areas include:

- graded algebras
- Clifford-like binary operator systems
- signed graph theory
- chiral decompositions
- bipartite spectral graph structures
- noncommutative transport systems
- parity-preserving and parity-breaking operators
- discrete gauge-like flows
- operator splitting methods

Double commutator structure

[H,[H,D]] = 2(D-HDH)

acts as a diffusion / smoothing operator between spectral sectors and may be viewed as:

- a discrete Laplacian-like object,
- or a spectral imbalance operator.

Key conceptual viewpoint

The Walsh–Hadamard system is not merely:

- local structure plus mixing,

but rather:

- local binary geometry interacting with
- global binary spectral geometry.